Mathematical Modeling of Street Light, Delhi, India

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Abstract

In this paper, we focus on mathematical model of street light. We determine the area traversed by the point source of light and then physically verify our generated model. In addition to that, we analyze some special cases our model to study the effect of angle of inclination on the area covered.

1 Introduction

We believe that the lights clamped on the streets should be adjusted in such a way that they cover the maximum of the road area. Hence, we are motivated to use our knowledge of simple calculus and apply it to solve a frequently encountered problem of lightning on streets.

In this model street light is assumed to be a point source of light, clamped at height h, inclined at an angle (θ) with the vertical. Now, light emerging out of it would be in the form of cone of solid angle (α) and our aim would be to establish to know the area swept on the ground by our source of light. To extend our model, we would be finding constraints on alpha and theta to verify our generated physical model.

2 Description of physical model

While generating this model, firstly, It is assumed that the light source is a point source of light which is attached at the height $(0, 0, h)$ (the axis on which the light is attached, assumed to be z axis), inclined at an angle (θ) with the vertical axis (z axis), and the light is falling in the xy plane. So, area of the light will be traced in the xy plane.

Secondly, the light from the right circular cone would emerge in the form of a right circular cone.

Thirdly, the solid angle (α) of the cone is taken to be constant for a given light. The lines OC and OB in figure 1 emerges out of the point source of the light & the ray OA is the axes of the cone , such that they intersect the x-axes at the points C', B and A' respectively.

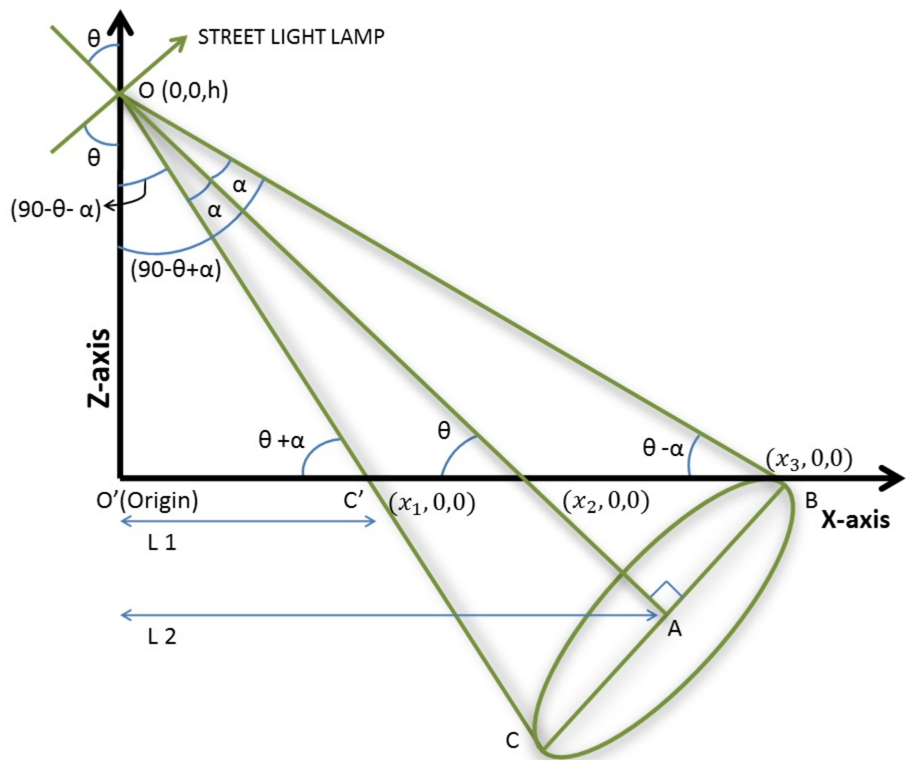

Figure 1: Mathematical Diagram

It is assumed that the rays are in the xz plane. $CABC$ is the base of the cone (it is the circular region of the cone). See figure 1

So here,

$$L_1 = x_1 = \frac{h}{tan(\theta + \alpha)}$$

$$L_2 = x_2 = \frac{h}{tan(\theta)}$$

$$L_3 = x_3 = \frac{h}{tan(\theta - \alpha)}$$

$In \Delta AOB$

$$sin\alpha = \frac{r}{y}$$

$In \Delta OO'B$

$$sin(\theta - \alpha) = \frac{h}{y}$$

$$y = \frac{h}{sin(\theta - \alpha)}$$

so,

$$sin\alpha = \frac{r}{h} sin(\theta - \alpha)$$

$$r = \frac{(h.sin\alpha)}{sin(\theta - \alpha)}$$
$$OA' = \frac{h}{sin\theta}$$
$$OC' = \frac{h}{sin(\theta + \alpha)}$$

Now,

$$y_1 = y_2 = y_3 = 0$$

OA',OC',OB are on the x-axis.

Co-ordinate of point B are
$$\frac{h}{tan(\theta - \alpha, 0, 0)}$$
Co-ordinate of point A' are
$$\frac{h}{tan(\theta, 0, 0)}$$
Co-ordinate of point C' are
$$\frac{h}{tan(\theta + \alpha, 0, 0)}$$

Whenever we put the light at an angle ,then the maximum length covered on the road by light would be
$$x = [\frac{h}{tan(\theta - \alpha)} - \frac{h}{tan(\theta + \alpha)}]$$

We already know, from our basic knowledge of calculus that the equation of the right circular cone whose vertex is at the point (α, β, γ) and whose axis is the line

EQUATION OF THE AXIS:
$$\frac{(x-\alpha)}{l} = \frac{(y-\beta)}{m} = \frac{(z-\gamma)}{n}$$

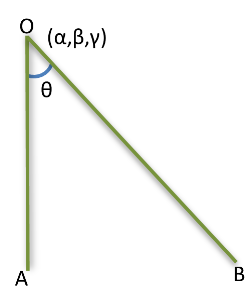

Figure 2: Mathematical Diagram

and semi-vertical angle θ is
$$[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2 = (l^2 + m^2 + n^2)cos^2\theta[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2] \quad (1)$$

3 Solution Procedures

So, in our case the equation of the axis of the cone is:

$$\frac{x-0}{\frac{h}{tan\theta}} = \frac{y-0}{0} = \frac{z-h}{-h}$$

The equation of our cone is,

$$\left[\frac{h}{tan\theta}x - h(z-h)\right]^2 = \left[\frac{h^2}{tan^2\theta} + h^2\right] cos^2\alpha \left(x^2 + y^2 + (z-h)^2\right) \tag{2}$$

Since, the curve is formed in the xy plane for that we put $z = 0$,

$$\left[\frac{h}{tan\theta}x + h^2\right]^2 = \left[\frac{h^2}{tan^2\theta} + h^2\right] cos^2\alpha \left(x^2 + y^2 + h^2\right) \tag{3}$$

$$\left(\frac{h^2}{tan^2\theta} - \frac{h^2}{tan^2\theta}cos^2\alpha - h^2cos^2\alpha\right)x^2 - h^2\left(\frac{1+tan^2\theta}{tan^2\theta}\right)cos^2\alpha y^2 + \frac{2h^3}{tan\theta}x = h^4\left(\frac{cos^2\alpha}{tan^2\theta} + cos^2\alpha - 1\right)$$

$$x^2\left(\frac{sin^2\alpha}{tan^2\theta} - cos^2\alpha\right) - \left(cosec^2\theta cos^2\alpha\right)y^2 + \frac{2hx}{tan\theta} = h^2\left(\frac{cos^2\alpha}{tan^2\theta} - sin^2\alpha\right) \tag{4}$$

Comparing the equation (4), with the standard equation of the conic section.

$$ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0$$

Where,

$$a = \left(\frac{sin^2\alpha}{tan^2\theta} - cos^2\alpha\right)$$

$$b = cosec^2\theta cos^2\alpha, h = 0$$

Evaluating,

$$\begin{aligned}
\left(h^2 - 4ab\right) &= 0 - \left[4\left(\frac{sin^2\alpha}{tan^2\theta} - cos^2\alpha\right)\left(-(cosec^2\theta.cos^2\alpha)\right)\right] \\
&= 4\left(\frac{sin^2\alpha}{tan^2\theta} - cos^2\alpha\right)\left(cosec^2\theta.cos^2\alpha\right) \\
&= 4\frac{sin^2\alpha.cos^2\alpha.cos^2\theta}{sin^4\theta} - 4cosec^2\theta.cos^4\alpha \\
&= 4\frac{cos^2\alpha}{sin^2\theta}\left[\frac{sin^2\alpha.cos^2\alpha}{sin^2\theta} - cos^2\theta\right] \\
&= 4\frac{cos^2\alpha}{sin^4\theta}\left[sin^2\alpha.cos^2\alpha - cos^2\theta.sin^2\theta\right] \\
&= 4\frac{cos^2\alpha}{sin^4\theta}\left[sin\alpha.cos\theta + cos\alpha.sin\theta\right]\left[sin\alpha.cos\theta - cos\alpha.sin\theta\right] \\
&= 4\frac{cos^2\alpha}{sin^4\theta}sin(\alpha+\theta).sin(\alpha-\theta)
\end{aligned}$$

If, $\alpha < \theta$ and $\mathbf{H^2 - 4AB < 0}$ then the region must be an Ellipse.

Dividing equation (5), by $sin^2\theta$, on both the sides

$$x^2\left(\frac{sin^2\alpha}{tan^2\theta} - cos^2\alpha\right) + \frac{2hx}{tan\theta} - \frac{cos^2\alpha}{sin^2\theta}y^2 = \frac{h^2}{sin^2\theta}\left[cos^2\alpha.cos^2\theta - sin^2\alpha.sin^2\theta\right]$$

$$x^2(sin^2\alpha.cos^2\theta - cos^2\alpha.sin^2\theta) + 2hx.sin\theta.cos\theta - cos^2\alpha y^2 = h^2cos(\alpha+\theta)cos(\alpha-\theta)$$

$$x^2sin(\alpha+\theta)sin(\alpha-\theta) + hx.sin2\theta - cos^2\alpha y^2 = h^2cos(\alpha+\theta)cos(\alpha-\theta)$$

$$h^2cos(\alpha+\theta)cos(\alpha-\theta) = x^2sin(\alpha+\theta)sin(\theta-\alpha) - h.x.sin2\theta + cos^2\alpha y^2 \quad (5)$$

$$= \left(\sqrt{sin(\alpha+\theta)sin(\theta-\alpha)}x\right)^2 - 2\left(\frac{hsin2\theta}{2}\right)\left(\frac{\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}{\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right)$$
$$+\left(\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta)sin(\theta-\alpha)}}\right) - \left(\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta)sin(\theta-\alpha)}}\right) + cos^2\alpha y^2$$

$$\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2 - h^2cos(\alpha+\theta).cos(\theta-\alpha)$$
$$= \left[\left(\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}\right)x - \frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2 + cos^2\alpha y^2$$

$$\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2 - h^2cos(\alpha+\theta).cos(\theta-\alpha)$$
$$= \left[\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}\right]^2\left[x - \frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2 + cos^2\alpha y^2 \quad (6)$$

$$\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2 - h^2cos(\alpha+\theta).cos(\theta-\alpha)$$
$$= \left[(x - \frac{hsin2\theta}{2sin(\alpha+\theta)sin(\theta-\alpha)})^2 + \frac{cos^2\alpha y^2}{sin(\alpha+\theta)sin(\theta-\alpha)}\right]\left[\sqrt{sin(\alpha+\theta)sin(\theta-\alpha)}\right]^2$$

$$\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2 - h^2cos(\alpha+\theta).cos(\theta-\alpha)}{sin(\alpha+\theta)sin(\theta-\alpha)}$$
$$= \left[x - \frac{hsin2\theta}{2sin(\alpha+\theta)sin(\theta-\alpha)}\right]^2 + \frac{cos^2\alpha y^2}{sin(\alpha+\theta)sin(\theta-\alpha)}$$

$$\frac{\left[x-\frac{hsin2\theta}{2sin(\alpha+\theta)sin(\theta-\alpha)}\right]^2}{\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha)}{sin(\alpha+\theta)sin(\theta-\alpha)}}+\frac{\frac{cos^2\alpha y^2}{sin(\alpha+\theta)sin(\theta-\alpha)}}{\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha)}{sin(\alpha+\theta)sin(\theta-\alpha)}}=1$$

$$\frac{\left[x-\frac{hsin2\theta}{2sin(\alpha+\theta)sin(\theta-\alpha)}\right]^2}{\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha)}{sin(\alpha+\theta)sin(\theta-\alpha)}}+\frac{y^2}{\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha))(sin(\alpha+\theta)sin(\theta-\alpha))}{[sin(\alpha+\theta)sin(\theta-\alpha)]cos^2\alpha}}=1$$

$$\frac{\left[x-\frac{hsin2\theta}{2sin(\alpha+\theta)sin(\theta-\alpha)}\right]^2}{\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha)}{sin(\alpha+\theta)sin(\theta-\alpha)}}+\frac{y^2}{\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha)}{cos^2\alpha}}=1$$

Which is analogous to equation of a Standard ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Where,

$$a^2=\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha)}{sin(\alpha+\theta)sin(\theta-\alpha)}$$

$$b^2=\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha)}{cos^2\alpha}$$

So,

$$a=\sqrt{\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha)}{sin(\alpha+\theta)sin(\theta-\alpha)}}$$

$$b=\sqrt{\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha)}{cos^2\alpha}}$$

Now,

$$\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha)}{sin(\alpha+\theta)sin(\theta-\alpha)}>0$$

Now,

$$\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2>0$$

And since, $90^\circ < cos(\alpha+\theta) < 180^\circ$
So,

$$(-h^2cos(\alpha+\theta)cos(\theta-\alpha))>0$$

Now,

$$\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2-h^2cos(\alpha+\theta).cos(\theta-\alpha)}{sin(\alpha+\theta)sin(\theta-\alpha)}>0$$

Would be greater than zero, only if

$$sin(\alpha+\theta)sin(\theta-\alpha)>0$$

Now,

$$(\alpha+\theta)<180$$

So,

$$sin(\alpha+\theta)>0$$

Now,

$$sin(\theta-\alpha)>0$$

would be greater than zero if,

$$\theta>\alpha$$

Now, we need to prove that

$$\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2 > h^2cos(\alpha+\theta)cos(\theta-\alpha)$$

$$\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2 > cos(\alpha+\theta)cos(\theta-\alpha)$$

Now,

$$sin(\alpha+\theta)sin(\theta-\alpha) > 0$$

$$sin^2 2\theta > 4cos(\alpha+\theta)cos(\theta-\alpha)sin(\alpha+\theta)sin(\theta-\alpha)$$

$$sin^2 2\theta > sin2(\alpha+\theta)sin2(\theta-\alpha)$$

AREA OF AN ELLIPSE:

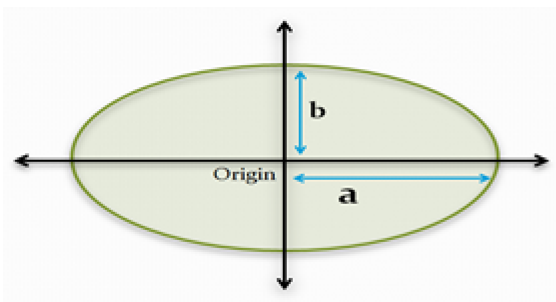

Figure 3: Mathematical Diagram

we know area of an ellipse $= \pi ab$
The above equations analogous to

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Where,

$$a = \sqrt{\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2 - h^2cos(\alpha+\theta).cos(\theta-\alpha)}{sin(\alpha+\theta)sin(\theta-\alpha)}}$$

and

$$b = \sqrt{\frac{\left[\frac{hsin2\theta}{2\sqrt{sin(\alpha+\theta).sin(\theta-\alpha)}}\right]^2 - h^2cos(\alpha+\theta).cos(\theta-\alpha)}{cos^2\alpha}}$$

Write the calculated area in the form of πab. Now since we know each light has its own angle alpha (α).

4 Physical Verification

FOR CALCULATING THE ANGLE ALPHA

In order to calculate the angle alpha, we took 3 different heights. Now, in that case we recorded the height of the light & found out the rough estimate of the radius of the circle traced, with which after applying the basic rules of trigonometry, we calculated the value of alpha.

CALCULATION OF ANGLE (α)

S.no	Height (cm)	Radius (cm)	Angle ($\alpha degree$)
1.	6	5	40.0
2.	8.5	8	43.22
3.	3‘	3.25	47.20
Average of the above three			44.00

CALCULATIONS INVOLVED

S.no		Angle (θ)	a (cm)	b (cm)	h (cm)	Area (A=πab)	By Formula
1.	Circle	90	7.75	7.75		188.59	185.205
2.	Ellipse	50	23/2	20/2	8	361.10	440.62
3.	Ellipse	46	67/2	43/2	12	2261.585	1806.81
4.	Ellipse	46	80/2	65/2	17	4082	3630.208

OBSERVATIONS

Figure 4: Physical Model

- Whenever the light would be inclined at an angle 90° with the vertical, the area thus traced would be of circle.

- Other than 90° with the vertical, the area thus traced would be an ellipse.
- The area traced would be an Ellipse, the angle theta (θ) with the pole of street light is greater than the semi-vertical angle alpha (α) of the cone generated.
- The area that traced on the ground to be positive, the angle theta (θ) with the pole of street light is greater than the semi-vertical angle alpha (α) of the cone generated.

SOURCES OF ERRORS

While calculating the semi-vertical angle alpha (α) of the cone and the area traced by the light on ground we take approximate measurement.